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Woven Strip Ornaments

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Abstract. A classification of strip ornaments is given, based on the discrete groups of symmetry mappings of a three-dimensional (arbitrarily) thin strip of infinite length and with a rectangular cross section.

Key Words: Strip ornaments,

MSC 2000: 51F15

1. Introduction

There are not only real textiles and wickerwork at baskets and fences. Also plane ornaments often try to provide the impression of being woven or twisted. If we want not to ignore this, once more arises the classification problem for the possible symmetries, as it has been done for plane ornaments long ago, [1, 5] etc. and recently [3, 4]. Beside the concrete interest, the study of woven ornaments seemingly is one possibility (next to Hausdorff's famous broken dimension) of doing geometry properly between plane and space. In the case of strip ornaments the domain of the symmetry mappings has to be a three-dimensional (arbitrarily) thin strip of infinite length and with a rectangular cross section (see Fig. 1).

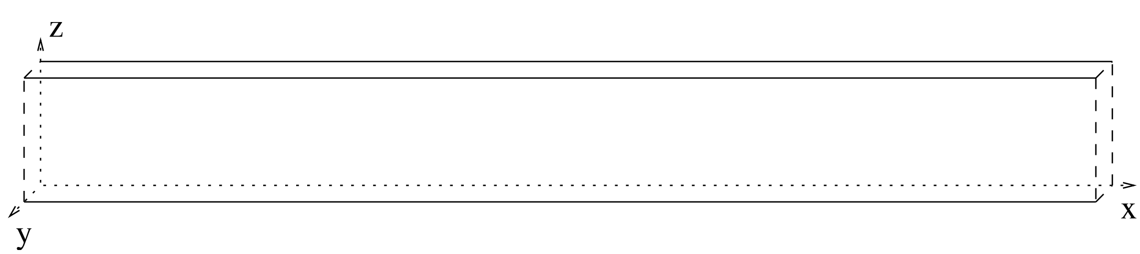

Figure 1: Strip

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2. The classification problem

We have to count all types of subgroups of the group G, generated by a minimal translation **t** in the direction x of the strip, reflections **rfy**, **rfz** with respect to the midplanes py (orthogonal to y-axis), pz (orthogonal to z-axis) of the strip, and a reflection **rfx** with respect to any plane orthogonal to the x-direction of the strip. As **t** is the product of two such reflections, a single **rfx** in common with **t** produces an infinity of reflections with respect to parallel planes orthogonal to the strip and with a pairwise minimal distance half of the length of minimal **t**. A simplified group table of full G (where e.g. $\mathbf{t} \circ \mathbf{t} = \mathbf{t}$ means that the product of two arbitrary translations in the x-direction is a translation of the same kind) can be seen in Table 1.

∘	rfx	rfy	rfz	rox	roy	roz	ps	trfy	trfz	s	t
t	**rfx**	**trfy**	**trfz**	**s**	**roy**	**roz**	**ps**	**trfy**	**trfz**	**s**	**t**
s	**ps**	**trfz**	**trfy**	**t**	**roz**	**roy**	**rfx**	**trfz**	**trfy**	**t**	
trfz	**roy**	**s**	**t**	**trfy**	**rfx**	**ps**	**roz**	**s**	**t**		
trfy	**roz**	**t**	**s**	**trfz**	**ps**	**rfx**	**roy**	**t**			
ps	**s**	**roy**	**roz**	**rfx**	**trfy**	**trfz**	**t**				
roz	**trfy**	**rfx**	**ps**	**roy**	**s**	**t**					
roy	**trfz**	**ps**	**rfx**	**roz**	**t**						
rox	**ps**	**rfz**	**rfy**	**id**							
rfz	**roy**	**rox**	**id**								
rfy	**roz**	**id**									
rfx	**id**										

Table 1: Abbreviated group table

In addition, **id** denotes the identity in this table and **ps** the point symmetry. **rox**, **roy**, **roz** denote the 180°-rotations around x-, y-, z-axis (note that there is only one **rox** but an infinity of **roy** and **roz**!). **trfy**, **trfz** denote the product of any **t** and the reflection in the midplanes py, pz, respectively. Finally **s** denotes 180°-screwings around the x-axis. Fig. 2 shows the position of any flag **id** after executing one of the described mappings.

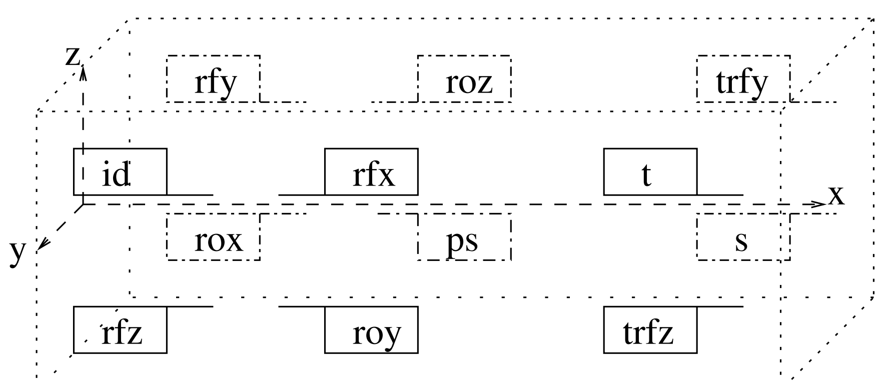


Figure 2: Mappings

Clearly two such discrete subgroups of full G are considered to be equivalent (of the same type) iff they are conjugated by an affinity, preserving the domain of the strip. Computer-aided searching for all subsets of the set $\{\mathbf{t}, \mathbf{s}, \mathbf{trfz}, \mathbf{trfy}, \mathbf{ps}, \mathbf{roz}, \mathbf{roy}, \mathbf{rox}, \mathbf{rfz}, \mathbf{rfy}, \mathbf{rfx}, \mathbf{id}\}$, closed with respect to the operation defined by Table 1, produces 33 possible cases, from which two are excluded to be subgroups by the following

Proposition 1. If any subgroup contains **trfy** and **trfz**, then it also contains **rfy**, **rfz** or **rox**.

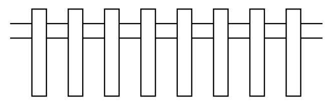
Proof: Let $\mathbf{t}_0$ be a translation generating the subgroup of translations, $\mathbf{t}_y$ the translation part of a minimal **trfy** and $\mathbf{t}_z$ the translation part of a minimal **trfz**. Then we have to distinguish the following three cases.

1. If $\mathbf{t}_0 = \mathbf{t}_y$ then $\mathbf{t}_0^{-1} \circ \mathbf{trfy} = \mathbf{t}_0^{-1} \circ \mathbf{t}_0 \circ \mathbf{rfy} = \mathbf{rfy}$.
2. If $\mathbf{t}_0 = \mathbf{t}_z$ then $\mathbf{t}_0^{-1} \circ \mathbf{trfz} = \mathbf{rfz}$.
3. And at last, if $\mathbf{t}_0 \neq \mathbf{t}_y$ and $\mathbf{t}_0 \neq \mathbf{t}_z$ then, because $\mathbf{trfy}^2$ is a translation, $\mathbf{t}_y^2 = \mathbf{t}_z^2 = \mathbf{t}_0$, hence $\mathbf{t}_y = \mathbf{t}_z = \frac{1}{2}\mathbf{t}_0$. We have $(\mathbf{t}_y \circ \mathbf{rfy})\,(\mathbf{t}_z \circ \mathbf{rfz})^{-1} = \mathbf{t}_y \circ \mathbf{t}_z^{-1} \circ \mathbf{rfy} \circ \mathbf{rfz} = \mathbf{rox}$. Square

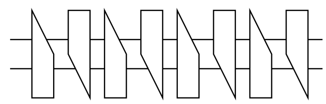
The remaining 31 cases list all types of subgroups. (In comparison: The usual classification of plane strip symmetries gives 7 types.) The same result of course could be obtained also by more extended use of the theory of discrete groups, [4, 5]. The 31 groups are listed below and each of them is illustrated by a simple example in the shape of a "fence".

1. **id**, **t**

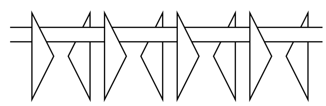
2. **id**, **rfx**, **t**



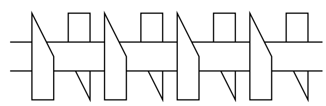
3. **id**, **roy**, **t**



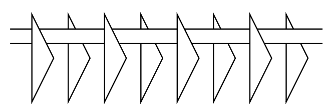
4. **id**, **roz**, **t**



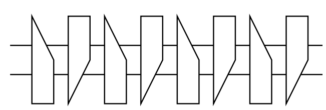
5. **id**, **ps**, **t**



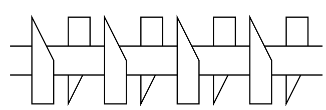
6. **id**, **trfy**, **t**



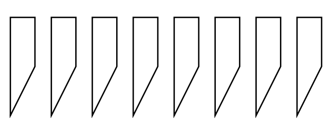
7. **id**, **trfz**, **t**



8. **id**, s, **t**

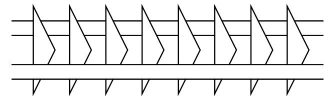


9. **id**, **rfy**, **trfy**, **t**

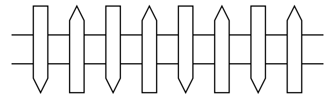


10. **id**, **rfz**, **trfz**, **t**

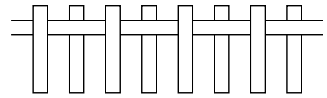
11. **id**, **rox**, s, **t**



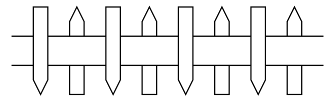
12. **id**, **rfx**, **roy**, **trfz**, **t**



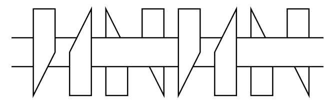
13. **id**, **rfx**, **roz**, **trfy**, **t**



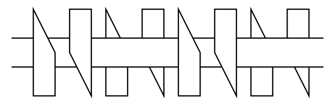
14. **id**, **rfx**, **ps**, s, **t**



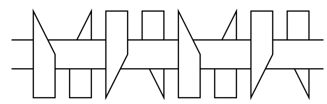
15. **id**, **roy**, **roz**, s, **t**



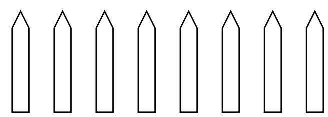
16. **id**, **roy**, **ps**, **trfy**, **t**



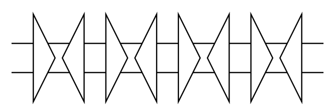
17. **id**, **roz**, **ps**, **trfz**, **t**



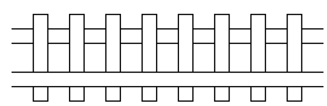
18. **id**, **rfx**, **rfy**, **roz**, **trfy**, **t**



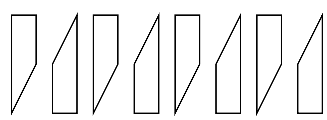
19. **id**, **rfx**, **rfz**, **roy**, **trfz**, **t**



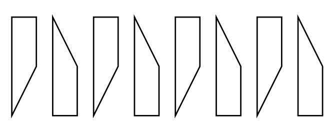
20. **id**, **rfx**, **rox**, **ps**, s, **t**



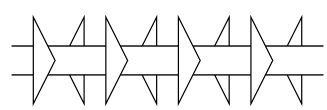
21. **id**, **rfy**, **roy**, **ps**, **trfy**, **t**



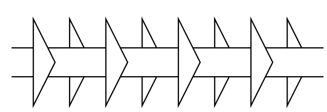
22. **id**, **rfy**, **trfy**, **trfz**, s, **t**



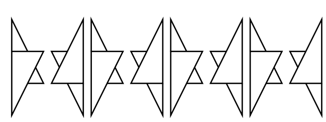
23. **id**, **rfz**, **roz**, **ps**, **trfz**, **t**



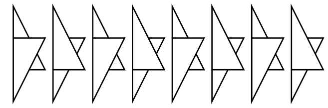
24. **id**, **rfz**, **trfy**, **trfz**, s, **t**



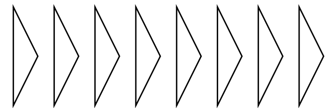
25. **id**, **rox**, **roy**, **roz**, s, **t**



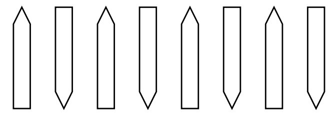
26. **id**, **rox**, **trfy**, **trfz**, **s**, **t**



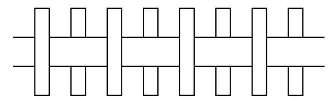
27. **id**, **rfy**, **rfz**, **rox**, **trfy**, **trfz**, **s**, **t**



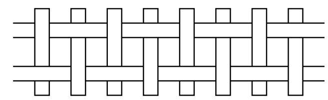
28. **id**, **rfx**, **rfy**, **roy**, **roz**, **ps**, **trfy**, **trfz**, **s**, **t**



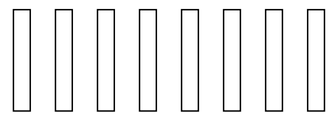
29. **id**, **rfx**, **rfz**, **roy**, **roz**, **ps**, **trfy**, **trfz**, **s**, **t**



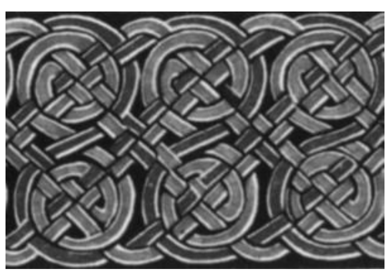
30. **id**, **rfx**, **rox**, **roy**, **roz**, **ps**, **trfy**, **trfz**, **s**, **t**



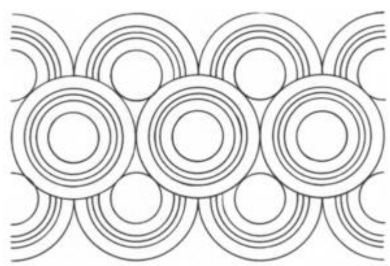
31. **id**, **rfx**, **rfy**, **rfz**, **rox**, **roy**, **roz**, **ps**, **trfy**, **trfz**, **s**, **t**



We conclude the note by some more interesting examples from historical decorations, each with its number from our list. Some of them are indeed plane woven ornaments ("layers"), but each such ornament, if permitting translations in at least one direction, may be considered also as a strip, of infinite extension in the z-direction or arbitrarily bounded by two appropriate parallels to the x-axis. As in the case of the seven types of plane strip ornaments, some of the 31 possibilities have been used frequently and in different cultures, but some others are very rare or seemingly not discovered by the designers.



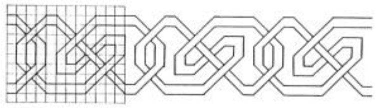
Border from a gospel-book,
Dublin, 7th century, No. 15



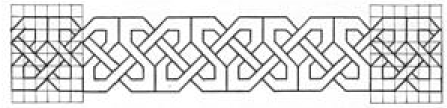
Shah Jehan Mosque,
Woking, 1889, No. 19

Floor in the baptistery
in Pisa, 12th century, No. 25

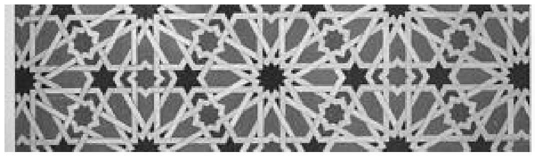
Renaissance beer-jug,
Lacroix et Serré, No. 25



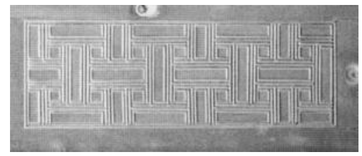
Stucco design, Alhambra Palace, Granada, No. 3



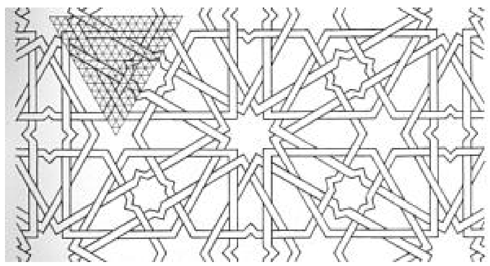
Stucco design, Alhambra Palace, Granada, No. 15



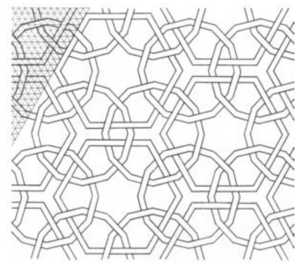
The Alcazar, Seville, No. 25



Süleymaniye Mosque, Istanbul, No. 29



The Alcazar, in detail



Hagia Sophia, Istanbul, No. 11

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