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Calculus of Variations

Symplectic Yang–Mills theory, Ricci tensor, and connections

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Abstract A Yang–Mills theory in a purely symplectic framework is developed. The corresponding Euler–Lagrange equations are derived and first integrals are given. We relate the results to the work of Bourgeois and Cahen on preferred symplectic connections.

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1 Introduction

The study of connections in vector resp. principal fiber bundles over manifolds has been a broad field of research in many contexts over decades. In particular, one is interested in properties of connections respecting various types of geometries.

In Riemannian geometry, we have the classical result that each Riemannian manifold (M, g) admits a distinguished linear connection, called the Levi–Civita connection. This connection is characterized by the conditions that it is metric, i.e. g is parallel, and that its torsion vanishes. More general, for any 2-form ζ on M with values in the tangent bundle TM of M, there exists a unique metric connection on (M, g) having ζ as its torsion.

Turning one's interest to symplectic geometry, the situation changes drastically. Namely, if (M, ω) is an almost symplectic manifold, then, for a given 2-form ζ on M

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with values in TM, the space of symplectic connections whose torsion is ζ is either empty or an infinite dimensional affine space (see Lemma 2.3). Here, by a symplectic connection, we mean the analog of a metric connection on a Riemannian manifold. Accordingly, a symplectic connection on (M, ω) is a linear connection ∇ on M such that $\nabla\omega = 0$.

By the above, on an almost symplectic manifold, there is no canonical symplectic connection singled out by any torsion condition, i.e. there is no notion analogous to the Levi–Civita connection. This fact gives rise to the question whether it is possible to select a single class of symplectic connections in other ways. In recent years, this problem has been approached by studying symplectic connections that, in addition to torsion-freeness, satisfy suitable curvature conditions. For an overview, see [2] and the references therein. In [4], the authors derived such a condition by a variational principle using a Lagrangian density with a quadratic polynomial in the curvature. The connections satisfying the deduced Euler–Lagrange equations are now referred to as preferred symplectic connections. Unfortunately, aside from the surface case, there is not much known about general properties of those connections. Progress has been made towards understanding so-called symplectic connections of Ricci type, which are described by a curvature condition that implies the Euler–Lagrange equations of [4]. But all these studies do not consider possible effects caused by the torsion. It is generally assumed that all connections taken into consideration are torsion-free.

It is the aim of the present work to develop an approach that also includes symplectic connections with non-vanishing torsion. Treating symplectic connections without any torsion obstruction has several advantages and is motivated by the following aspects.

Also in Riemannian geometry, in spite of the existence of the Levi–Civita connection, one is more and more interested in connections with torsion. This is due to current issues in string theory. There one studies different types of metric connections having good but non-trivial torsion. Here, in most cases, “good” means that the torsion considered as a covariant 3-tensor field is totally skew-symmetric (see e.g. [1]). Another point is that dropping the restriction to torsion-free connections allows to take into account connections that are Hermitian with respect to a compatible almost complex structure. In general, those connections have non-trivial torsion. On the other hand, for fixed compatible almost complex structure, there exists a distinguished Hermitian connection (see [9]). A further reason comes from the theory of symplectic Dirac operators. The more one studies these operators, the more one becomes convinced that, at least in this context, certain symplectic connections with torsion are more suitable than torsion-free connections (see [10]). Moreover, only admitting symplectic connections with torsion makes it possible to extend the considerations to almost symplectic manifolds (cp. Proposition 2.5).

The approach given here has the advantage that it also works for connections in vector bundles. Generalizing the ansatz in [4], it gives a purely symplectic Yang–Mills theory. At this point, we should mention the paper [13]. At first glance, this paper is closely related to that presented here, and really there are some relations to our work. However, Urakawa studies the usual Yang–Mills functional, using the symplectic framework. More precisely, he considers the variation of the integral

$$\int_M g\left(R^{\nabla}, R^{\nabla}\right) \omega^n ,$$

where g is a compatible Riemannian metric on the symplectic manifold (M, ω). Starting from the Kähler case, Urakawa reformulates classical Yang–Mills theory and investigates it in terms of symplectic geometry, provided that the Riemannian metric the theory is based on is associated to a symplectic structure on the underlying manifold. In particular, this approach is not applicable to symplectic connections on a symplectic manifold without fixing an auxiliary structure. The point of view developed here is completely different. We present an intrisically symplectic Yang–Mills theory which directly handles the given symplectic data. The price we have to pay for this is that the analytic nature of the Yang–Mills functional introduced here is more complicated than that of the functional studied by Urakawa. For example, the Lagrange density of the second functional is always non-negative, which is not the case for our one.

Dealing with Yang–Mills theory on an almost symplectic manifold, the so-called symplectic Ricci tensor comes naturally into play. This tensor field generalizes the usual Ricci tensor for a torsion-free symplectic connection as well as the $*$-Ricci tensor of an almost Hermitian manifold. Furthermore, it has the advantage that the corresponding endomorphism, called the symplectic Ricci operator, can be defined also for connections in a vector bundle.

The paper is organized as follows. In the second and third section, we recall some results on the torsion of metric and symplectic connections and on Ricci tensors, respectively. In Sect. 4, we introduce two functionals. The first one is the symplectic analog of the Yang–Mills functional, whereas the second one is defined by the symplectic Ricci operator. We compute their Euler–Lagrange equations and describe how they are related. In Sect. 5, we discuss first integrals of the symplectic Yang–Mills equation, which, in dimension 4, are symplectic analogs of the self-duality and anti-self-duality equation. In the last section, we relate our results to that in [4]. In particular, we give a proof for the Euler–Lagrange equations of the functionals considered there.

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2 Connections and torsion

In this section, we summarize some well known facts on the torsion of metric and symplectic connections.

Let M be a smooth manifold. If E is a vector bundle over M, we denote the space of smooth sections of E by $\Gamma(E)$, the space of smooth k-forms on M with values in E by $\Omega^k(M, E)$, and the endomorphism bundle of E by $\mathrm{End}(E)$.

Let $\mathcal{C}(M)$ be the space of linear connections on M, i.e. of connections in the tangent bundle TM of M. Then $\mathcal{C}(M)$ is an affine space over the vector space $\Omega^1(M, \mathrm{End}(TM))$. The torsion $T^\nabla \in \Omega^2(M, TM)$ of a connection $\nabla \in \mathcal{C}(M)$ is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for $X, Y \in \Gamma(TM)$. Thus the map $\nabla \in \mathcal{C}(M) \mapsto T^\nabla \in \Omega^2(M, TM)$ is affine and the associated linear map $\Phi : \Omega^1(M, \mathrm{End}(TM)) \to \Omega^2(M, TM)$ is given by

$$\Phi(\theta)(X, Y) = \theta(X)Y - \theta(Y)X.$$

A connection $\nabla \in \mathcal{C}(M)$ is called torsion-free if $T^\nabla = 0$.

Let g be a Riemannian metric on M. A metric connection on M is a connection $\nabla \in \mathcal{C}(M)$ such that $\nabla g = 0$, i.e.

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. The space $\mathcal{C}(M,g)$ of metric connections on M is an affine subspace of $\mathcal{C}(M)$ and its corresponding vector space $\mathcal{E}^1(M,g)$ is formed by all $\theta \in \Omega^1(M, \mathrm{End}(TM))$ that satisfy

$$g(\theta(X)Y, Z) = -g(\theta(X)Z, Y). \tag{2.1}$$

Lemma 2.1 *The restriction of Φ to $\mathcal{E}^1(M,g)$ is an isomorphism onto the space $\Omega^2(M,TM)$.*

Proof Let $\theta \in \mathcal{E}^1(M,g)$ and suppose that $\Phi(\theta) = 0$, i.e.

$$\theta(X)Y = \theta(Y)X.$$

By means of Eq. (2.1), we conclude

$$\begin{aligned} g(\theta(X)Y,Z) &= -g(\theta(X)Z,Y) = -g(\theta(Z)X,Y) = g(\theta(Z)Y,X) \\ &= g(\theta(Y)Z,X) = -g(\theta(Y)X,Z) = -g(\theta(X)Y,Z)\,, \end{aligned}$$

which implies $\theta = 0$.

Let $\zeta \in \Omega^2(M,TM)$ and let $\theta \in \Omega^1(M, \mathrm{End}(TM))$ be determined by

$$2\,g(\theta(X)Y,Z) = g(\zeta(X,Y),Z) + g(\zeta(Z,X),Y) + g(\zeta(Z,Y),X).$$

One readily verifies that $\theta \in \mathcal{E}^1(M,g)$ and that

$$g(\theta(X)Y,Z) - g(\theta(Y)X,Z) = g(\zeta(X,Y),Z)\,,$$

which means $\Phi(\theta) = \zeta$. □

An immediate consequence is

Corollary 2.2 *The map $\nabla \in \mathcal{C}(M,g) \mapsto T^\nabla \in \Omega^2(M,TM)$ is $1:1$. In particular, there is a unique connection $\nabla \in \mathcal{C}(M,g)$, called the Levi–Civita connection, such that $T^\nabla = 0$.*

From now on, we suppose that M is endowed with an almost symplectic structure, i.e. a non-degenerate 2-form ω. The form ω is called a symplectic structure if in addition it is closed. A connection $\nabla \in \mathcal{C}(M)$ is said to be symplectic if $\nabla\omega = 0$, i.e. if

$$X(\omega(Y,Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. We point out that we do not require that a symplectic connection is torsion-free. The space $\mathcal{C}(M,\omega)$ of symplectic connections on M is again an affine subspace of $\mathcal{C}(M)$ and its vector space $\mathcal{E}^1(M,\omega)$ consists of all $\theta \in \Omega^1(M, \mathrm{End}(TM))$ that satisfy

$$\omega(\theta(X)Y,Z) = \omega(\theta(X)Z,Y).$$

Although a symplectic connection is the analog of a metric connection, the properties of the torsion maps are completely different (cf. [12]).

Lemma 2.3 (i) *A form $\theta \in \mathcal{E}^1(M,\omega)$ satisfies $\Phi(\theta) = 0$ if and only if the expression $\omega(\theta(X)Y, Z)$ is totally symmetric in $X, Y, Z \in \Gamma(TM)$.*

(ii) *The image $\Phi\left(\mathcal{E}^1(M,\omega)\right)$ of $\mathcal{E}^1(M,\omega)$ under Φ is the space of all $\zeta \in \Omega^2(M, TM)$ such that*

$$\omega(\zeta(X,Y),Z) + \omega(\zeta(Y,Z),X) + \omega(\zeta(Z,X),Y) = 0. \tag{2.2}$$

Proof Assertion (i) is obvious. Moreover, it is easy to check that Eq. (2.2) holds true for any $\zeta \in \Phi\left(\mathcal{E}^1(M,\omega)\right)$. Finally, suppose that $\zeta \in \Omega^2(M, TM)$ satisfies Eq. (2.2) and let $\theta \in \Omega^1(M, \mathrm{End}(TM))$ be given by

$$\omega(\theta(X)Y,Z) = \frac{1}{3}(\omega(\zeta(X,Y),Z) + \omega(\zeta(X,Z),Y)).$$

Then $\theta \in \mathcal{E}^1(M,\omega)$ and $\Phi(\theta) = \zeta$. □

Corollary 2.4 *The map $\nabla \in \mathcal{C}(M,\omega) \mapsto T^\nabla \in \Omega^2(M,TM)$ is neither injective nor onto. Moreover, the pre-image $\left\{\nabla \in \mathcal{C}(M,\omega) : T^\nabla = \zeta\right\}$ of a form $\zeta \in \Omega^2(M,TM)$ is either empty or infinite dimensional.*

A further difference to the metric case is

Proposition 2.5 *There exists a torsion-free symplectic connection ∇ on M if and only if ω is a symplectic structure.*

Proof This follows from the following two facts. For any $\nabla \in \mathcal{C}(M,\omega)$, we have

$$\mathrm{d}\omega(X,Y,Z) = \omega\left(T^\nabla(X,Y),Z\right) + \omega\left(T^\nabla(Y,Z),X\right) + \omega\left(T^\nabla(Z,X),Y\right).$$

If $\nabla^0 \in \mathcal{C}(M)$ is any torsion-free connection and $\theta \in \Omega^1(M, \mathrm{End}(TM))$ is defined by

$$\omega(\theta(X)Y,Z) = \frac{1}{3}\left(\left(\nabla^0_X\omega\right)(Y,Z) + \left(\nabla^0_Y\omega\right)(X,Z)\right),$$

then the connection $\nabla = \nabla^0 + \theta$ is also torsion-free and

$$X(\omega(Y,Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = \frac{1}{3}\mathrm{d}\omega(X,Y,Z).$$

□

According to the above, in symplectic geometry, there is no analog of the Levi–Civita connection. Furthermore, in the case that ω is a symplectic structure, any connection $\nabla \in \mathcal{C}(M,\omega)$ such that $\omega\left(T^\nabla(X,Y),Z\right)$ is symmetric or skew-symmetric in Y, Z has to be torsion-free.

3 Ricci tensors

In the following, let M have dimension $2n$ and let $\mathsf{s} = (\mathsf{e}_1, \ldots, \mathsf{e}_{2n})$ be a symplectic frame on some open subset $U \subset M$, i.e. a frame of vector fields on U such that

$$\omega(\mathsf{e}_i, \mathsf{e}_j) = \omega(\mathsf{e}_{n+i}, \mathsf{e}_{n+j}) = 0 \quad \text{and} \quad \omega(\mathsf{e}_i, \mathsf{e}_{n+j}) = \delta_{ij}$$

for $i, j = 1, \ldots, n$. Furthermore, let J^{s} be the almost complex structure on U defined by

$$J^{\mathsf{s}}\mathsf{e}_i = \mathsf{e}_{n+i}$$

for $i = 1, \ldots, n$.
Let ∇ be any connection on M. Its curvature is the form $R^{\nabla} \in \Omega^2(M, \mathrm{End}(TM))$ given by

$$R^{\nabla}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{3.1}$$

The Ricci tensor of ∇ is the tensor field $\mathrm{ric}^{\nabla} \in \Gamma(T^*M \otimes T^*M)$ defined by

$$\begin{aligned}\mathrm{ric}^{\nabla}(X,Y) &= \mathrm{Tr}\left(Z \mapsto R^{\nabla}(Z,X)Y\right)\\ &= \sum_{i=1}^{2n} \omega\left(R^{\nabla}(\mathsf{e}_i, X)Y, J^{\mathsf{s}}\mathsf{e}_i\right).\end{aligned}$$

Thus the Ricci tensor ric^{∇} is obtained by contracting the curvature R^{∇} with respect to the symplectic form ω. Following the ideas of Vaisman [14], we consider another contraction of R^{∇}. We define $\mathrm{sRic}^{\nabla} \in \Gamma(\mathrm{End}(TM))$ by

$$\mathrm{sRic}^{\nabla}(X) = \sum_{i=1}^{n} R^{\nabla}(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i)X$$

and call it the symplectic Ricci operator. Moreover, we define the symplectic Ricci tensor as the tensor field $\mathrm{sric}^{\nabla} \in \Gamma(T^*M \otimes T^*M)$ given by

$$\mathrm{sric}^{\nabla}(X,Y) = \omega\left(\mathrm{sRic}^{\nabla}(X), Y\right).$$

Remark 3.1 If ω is the Kähler form of an almost Hermitian structure (g, J) and ∇ is the Levi-Civita connection of g, then sric^{∇} is the so-called $*$-Ricci tensor. See e.g. [15].

The Ricci tensors ric^{∇} and sric^{∇} are related by

Proposition 3.2 *If $\nabla \in \mathcal{C}(M,\omega)$, then*

$$\mathrm{sric}^{\nabla}(X,Y) - \mathrm{ric}^{\nabla}(X,Y) = \omega\left(K^{\nabla}(X), Y\right),$$

where

$$\begin{aligned}K^{\nabla}(X) &= \sum_{i=1}^{n}\left(T^{\nabla}\left(T^{\nabla}(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i), X\right) + \left(\nabla_X T^{\nabla}\right)(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i)\right)\\ &\quad + \sum_{i=1}^{2n}\left(T^{\nabla}\left(T^{\nabla}(X, \mathsf{e}_i), J^{\mathsf{s}}\mathsf{e}_i\right) + \left(\nabla_{\mathsf{e}_i} T^{\nabla}\right)(J^{\mathsf{s}}\mathsf{e}_i, X)\right).\end{aligned}$$

Proof Let $\nabla \in \mathcal{C}(M,\omega)$. Then

$$\omega\left(R^{\nabla}(X,Y)Z_1, Z_2\right) = \omega\left(R^{\nabla}(X,Y)Z_2, Z_1\right). \tag{3.2}$$

With this, one gets

$$\begin{aligned}&\mathrm{sric}^{\nabla}(X,Y) - \mathrm{ric}^{\nabla}(X,Y)\\ &\quad = \sum_{i=1}^{n} \omega\left(R^{\nabla}(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i)X + R^{\nabla}(J^{\mathsf{s}}\mathsf{e}_i, X)\mathsf{e}_i + R^{\nabla}(X, \mathsf{e}_i)J^{\mathsf{s}}\mathsf{e}_i, Y\right).\end{aligned}$$

Now, applying the first Bianchi identity (cf. [11, Chapter III, Theorem 5.3]), the assertion follows. □

Consequently, for any torsion-free symplectic connection ∇, the Ricci tensors ric^∇ and sric^∇ coincide. But in general, for a generic symplectic connection, this is not true.

Proposition 3.3 *If* $\nabla \in \mathcal{C}(M,\omega)$, *then*

$$\mathrm{sric}^\nabla(X,Y) = \mathrm{sric}^\nabla(Y,X)$$

for any $X, Y \in \Gamma(TM)$.

Proof This easily follows from Eq. (3.2). □

By Proposition 3.2, in general, ric^∇ is not symmetric. This, among others, indicates that the symplectic Ricci tensor sric^∇ is more adapted to symplectic geometry than the usual Ricci tensor ric^∇. Furthermore, the symplectic Ricci operator sRic^∇ can be straightforwardly generalized to connections ∇ in a vector bundle on M. If E is a vector bundle on M and $\nabla : \Gamma(E) \to \Omega^1(M,E)$ is a connection in E, we define the symplectic Ricci operator $\mathrm{sRic}^\nabla \in \Gamma(\mathrm{End}(E))$ of ∇ by

$$\mathrm{sRic}^\nabla(\xi) = \sum_{i=1}^{n} R^\nabla(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i)\xi$$

for a section $\xi \in \Gamma(E)$, where the curvature $R^\nabla \in \Omega^2(M,\mathrm{End}(E))$ is given analogously to Eq. (3.1) by

$$R^\nabla(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi.$$

4 Symplectic Yang–Mills functionals

In this section, we generalize the variational principle for symplectic connections suggested by Bourgeois and Cahen [4] to connections in vector bundles.
We now suppose that the manifold M is closed. Let E be a real vector bundle over M of rank $2m$ with an almost symplectic structure, i.e. a non-degenerate 2-form $\mathsf{b} \in \Gamma\left(\Lambda^2 E\right)$. Let $\mathfrak{s} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_{2m})$ be a local symplectic frame in E and let $J^{\mathfrak{s}}$ be the local almost complex structure in E given by

$$J^{\mathfrak{s}}\mathfrak{e}_i = \mathfrak{e}_{m+i} \quad \text{for} \quad i = 1, \ldots, m.$$

We define pairings

$$(K,L) \in \Gamma(\mathrm{End}(E)) \times \Gamma(\mathrm{End}(E)) \mapsto \mathsf{b}(K,L) \in C^\infty(M)$$

and

$$(\alpha,\beta) \in \Omega^k(M,\mathrm{End}(E)) \times \Omega^k(M,\mathrm{End}(E)) \mapsto \mathsf{b}(\alpha,\beta) \in C^\infty(M)$$

by

$$\mathsf{b}(K,L) = \sum_{i=1}^{2m} \mathsf{b}(K\mathfrak{e}_i, LJ^{\mathfrak{s}}\mathfrak{e}_i)$$

and

$$\mathsf{b}(\alpha,\beta) = \sum_{1\le i_1<\cdots<i_k\le 2n} \mathsf{b}(\alpha(\mathsf{e}_{i_1},\ldots,\mathsf{e}_{i_k}),\beta(J^{\mathsf{s}}\mathsf{e}_{i_1},\ldots,J^{\mathsf{s}}\mathsf{e}_{i_k})).$$

It is easy to see that $\mathsf{b}(K,L)$ and $\mathsf{b}(\alpha,\beta)$ do not depend on the choice of the symplectic frames s and $\mathfrak{s}$ and that

$$\mathsf{b}(K,L) = \mathsf{b}(L,K) \quad \text{and} \quad \mathsf{b}(\alpha,\beta) = (-1)^k\mathsf{b}(\beta,\alpha).$$

Let $\mathcal{C}(E)$ denote the space of connections in E and let $\mathcal{C}(E,\mathsf{b})$ be the subspace of symplectic connections in E, i.e. of connections $\nabla \in \mathcal{C}(E)$ such that

$$X(\mathsf{b}(\xi_1,\xi_2)) = \mathsf{b}(\nabla_X\xi_1,\xi_2) + \mathsf{b}(\xi_2,\nabla_X\xi_1)$$

for all $X \in \Gamma(TM)$ and $\xi_1,\xi_2 \in \Gamma(E)$. Then $\mathcal{C}(E)$ is an affine space over the vector space $\Omega^1(M,\mathrm{End}(E))$ and $\mathcal{C}(E,\mathsf{b})$ is an affine subspace over the vector space $\mathcal{E}^1(E,\mathsf{b})$ of all $\theta \in \Omega^1(M,\mathrm{End}(E))$ that satisfy

$$\mathsf{b}(\theta(X)\xi_1,\xi_2) = \mathsf{b}(\theta(X)\xi_2,\xi_1).$$

The symplectic analog of the Yang–Mills functional is now the functional

$$I_1 : \mathcal{C}(E,\mathsf{b}) \to \mathbb{R}\,, \quad I_1(\nabla) = \frac{1}{2}\int_M \mathsf{b}\left(R^\nabla, R^\nabla\right)\omega^{(n)}.$$

Besides this, we consider the functional

$$I_2 : \mathcal{C}(E,\mathsf{b}) \to \mathbb{R}\,, \quad I_2(\nabla) = \frac{1}{2}\int_M \mathsf{b}\left(\mathrm{sRic}^\nabla, \mathrm{sRic}^\nabla\right)\omega^{(n)}.$$

Here we have used the abbreviation

$$\omega^{(k)} = \frac{1}{k!}\omega^k.$$

In particular, $\omega^{(n)}$ is the symplectic volume form.
To compute the Euler–Lagrange equations of the functionals I_1 and I_2, we need some preparations. Let $\Omega^k(M)$ denote the space of smooth k-forms on M and let

$$(\alpha,\alpha') \in \Omega^k(M,\mathrm{End}(E)) \times \Omega^l(M,\mathrm{End}(E)) \mapsto \mathsf{b}(\alpha\wedge\alpha') \in \Omega^{k+l}(M)$$

be the bilinear map determined by

$$\mathsf{b}((K\otimes\varphi)\wedge(L\otimes\psi)) = \mathsf{b}(K,L)\varphi\wedge\psi$$

for $K,L \in \Gamma(\mathrm{End}(E))$, $\varphi \in \Omega^k(M)$ and $\psi \in \Omega^l(M)$. The symplectic analog of the Riemannian Hodge operator is defined as follows (cf. [5]). The symplectic Hodge operator is the unique isomorphism $* : \Omega^k(M,\mathrm{End}(E)) \to \Omega^l(M,\mathrm{End}(E))$ that satisfies

$$\mathsf{b}(\alpha,\beta)\,\omega^{(n)} = \mathsf{b}(\alpha\wedge *\beta)$$

for any $\alpha,\beta \in \Omega^k(M,\mathrm{End}(E))$. In the next lemma, we note some properties of this operator (cf. [8]).

Lemma 4.1 (i) *For any $\alpha \in \Omega^k(M,\mathrm{End}(E))$ and any k, it is $*(*\alpha) = \alpha$.*

(ii) *For any* $L \in \Gamma(\mathrm{End}(E))$, *it is* $*L = L \otimes \omega^{(n)}$.
(iii) *If* $\alpha \in \Omega^1(M, \mathrm{End}(E))$, *then* $*\alpha = \alpha \wedge \omega^{(n-1)}$.
(iv) *If* $\alpha \in \Omega^2(M, \mathrm{End}(E))$, *then*

$$*\alpha = \alpha(\omega) \otimes \omega^{(n-1)} - \alpha \wedge \omega^{(n-2)}.$$

Here $\alpha(\omega) \in \Gamma(\mathrm{End}(E))$ for $\alpha \in \Omega^2(M, \mathrm{End}(E))$ means the section given by

$$\alpha(\omega) = \sum_{i=1}^{n} \alpha(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i).$$

If ∇ is a connection in E, let $\mathrm{d}^\nabla : \Omega^k(M, \mathrm{End}(E)) \to \Omega^{k+1}(M, \mathrm{End}(E))$ be the associated exterior differential and set

$$\delta^\nabla = (-1)^{k+1} * \mathrm{d}^\nabla * : \Omega^{k+1}(M, \mathrm{End}(E)) \to \Omega^k(M, \mathrm{End}(E)).$$

Then δ^∇ is the formal adjoint of d^∇ in the following sense.

Proposition 4.2 *Let* $\alpha \in \Omega^k(M, \mathrm{End}(E))$ *and* $\beta \in \Omega^{k+1}(M, \mathrm{End}(E))$. *Then, for any* $\nabla \in \mathcal{C}(E, \mathsf{b})$,

$$\int_M \mathsf{b}\left(\mathrm{d}^\nabla \alpha, \beta\right) \omega^{(n)} = \int_M \mathsf{b}\left(\alpha, \delta^\nabla \beta\right) \omega^{(n)}.$$

Proof One can proceed as in the Riemannian case. Since the connection ∇ is symplectic, we have

$$\mathrm{d}(\mathsf{b}(\alpha \wedge *\beta)) = \mathsf{b}\left(\mathrm{d}^\nabla \alpha \wedge *\beta\right) + (-1)^k \mathsf{b}\left(\alpha \wedge \mathrm{d}^\nabla *\beta\right).$$

Using in addition Stokes' Theorem and Lemma 4.1(i), we conclude

$$\begin{aligned}
\int_M \mathsf{b}\left(\mathrm{d}^\nabla \alpha, \beta\right) \omega^{(n)} &= \int_M \left(\mathrm{d}^\nabla \alpha \wedge *\beta\right) \omega^{(n)} \\
&= \int_M \mathrm{d}(\mathsf{b}(\alpha \wedge *\beta)) \omega^{(n)} - (-1)^k \int_M \mathsf{b}\left(\alpha \wedge \mathrm{d}^\nabla *\beta\right) \omega^{(n)} \\
&= (-1)^{k+1} \int_M \mathsf{b}\left(\alpha \wedge **\mathrm{d}^\nabla *\beta\right) \omega^{(n)} \\
&= \int_M \mathsf{b}\left(\alpha, \delta^\nabla \beta\right) \omega^{(n)}.
\end{aligned}$$

□

For the action of δ^∇ on 1-forms, we have (cp. also [5], Sect. 1.2 and Theorem 2.2.1)

Lemma 4.3 *Suppose that* ω *is a symplectic structure and let* ∇ *be any connection in* E. *Then*

$$\delta^\nabla \alpha = -\mathrm{d}^\nabla \alpha(\omega)$$

for $\alpha \in \Omega^1(M, \mathrm{End}(E))$.

Proof By means of Lemma 4.1, we derive

$$\begin{aligned}\delta^{\nabla} &= -*\mathrm{d}^{\nabla} * \alpha \\ &= -*\mathrm{d}^{\nabla}\left(\alpha \wedge \omega^{(n-1)}\right) \\ &= -*\left(\mathrm{d}^{\nabla}\alpha \wedge \omega^{(n-1)}\right) \\ &= -*\left(\mathrm{d}^{\nabla}\alpha(\omega) \otimes \omega^{(n)}\right) \\ &= -\mathrm{d}^{\nabla}\alpha(\omega).\end{aligned}$$

□

Proposition 4.4 *If ω is a symplectic structure, then the functionals I_1 and I_2 differ by a constant.*

Proof We can proceed as in [4], Sect. 2. First we observe that, by Lemma 4.1(iv),

$$\begin{aligned}\mathsf{b}(\alpha \wedge \beta) \wedge \omega^{(n-2)} &= \mathsf{b}\left(\alpha \wedge \left(\beta \wedge \omega^{(n-2)}\right)\right) \\ &= \mathsf{b}\left(\alpha \wedge \left(\beta(\omega) \otimes \omega^{(n-1)}\right)\right) - \mathsf{b}(\alpha \wedge *\beta) \\ &= \left(\mathsf{b}(\alpha(\omega), \beta(\omega)) - \mathsf{b}(\alpha, \beta)\right) \omega^{(n)}\end{aligned}$$

for all $\alpha, \beta \in \Omega^2(M, \mathrm{End}(E))$. In particular,

$$\left(\mathsf{b}\left(\mathrm{sRic}^{\nabla}, \mathrm{sRic}^{\nabla}\right) - \mathsf{b}\left(R^{\nabla}, R^{\nabla}\right)\right)\omega^{(n)} = \mathsf{b}\left(R^{\nabla} \wedge R^{\nabla}\right) \wedge \omega^{(n-2)}.$$

Hence, by $\mathrm{d}^{\nabla} R^{\nabla} = 0$, $\mathrm{d}\omega = 0$ and Stokes' Theorem, for ∇^t as in the proof of Theorem 4.5, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(I_2 - I_1)\left(\nabla^t\right)\Big|_{t=0} &= \int_M \mathsf{b}\left(\mathrm{d}^{\nabla}\theta \wedge R^{\nabla}\right) \wedge \omega^{(n-2)} \\ &= \int_M \mathrm{d}\left(\mathsf{b}\left(\theta \wedge R^{\nabla}\right) \wedge \omega^{(n-2)}\right) \\ &= 0.\end{aligned}$$

□

Now we can prove

Theorem 4.5 (i) *A connection $\nabla \in \mathcal{C}(E, \mathsf{b})$ is a critical point of the functional I_1 if and only if $\mathrm{d}^{\nabla} * R^{\nabla} = 0$.*

(ii) *In case ω is a symplectic structure, a connection $\nabla \in \mathcal{C}(E, \mathsf{b})$ is a critical point of the functional I_2 if and only if $\nabla \mathrm{sRic}^{\nabla} = 0$. Moreover, in this case, the Euler–Lagrange equations of I_1 and I_2 are equivalent.*

Proof Let $\theta \in \mathcal{E}^1(E, \mathsf{b})$ and let ∇^t be a smooth curve in $\mathcal{C}(E, \mathsf{b})$ with $\nabla^0 = \nabla$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla^t\Big|_{t=0} = \theta.$$

Then, as is well known,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} R^{\nabla^t}\right|_{t=0} = \mathrm{d}^\nabla \theta .$$

Thus, by means of Proposition 4.2, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} I_1\left(\nabla^t\right)\right|_{t=0} = \int_M \mathsf{b}\left(\mathrm{d}^\nabla\theta, R^\nabla\right)\omega^{(n)} = \int_M \mathsf{b}\left(\theta, \delta^\nabla R^\nabla\right)\omega^{(n)} .$$

Since $\delta^\nabla R^\nabla \in \mathcal{E}^1(E,\mathsf{b})$ because of the symplecticity of ∇ and since also the restriction of b to $\mathcal{E}^1(E,\mathsf{b})$ is non-degenerate, Assertion (i) follows.
Now we suppose that ω is a symplectic structure. Then we can apply Lemma 4.3 to obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{sRic}^{\nabla^t}\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t} R^{\nabla^t}(\omega)\right|_{t=0} = \mathrm{d}^\nabla\theta(\omega) = -\delta^\nabla\theta .$$

Hence

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} I_2\left(\nabla^t\right)\right|_{t=0} = -\int_M \mathsf{b}\left(\delta^\nabla\theta, \mathrm{sRic}^\nabla\right)\omega^{(n)} = -\int_M \mathsf{b}\left(\theta, \nabla\mathrm{sRic}^\nabla\right)\omega^{(n)} .$$

Since also $\nabla\mathrm{sRic}^\nabla \in \mathcal{E}^1(E,\mathsf{b})$, this gives the first part of Assertion (ii). The second part is a consequence of Proposition 4.4. Alternatively, it follows from

$$\begin{aligned}
\mathrm{d}^\nabla{*}R^\nabla &= \mathrm{d}^\nabla\left(R^\nabla(\omega)\otimes\omega^{(n-1)} - R^\nabla\wedge\omega^{(n-2)}\right)\\
&= \mathrm{d}^\nabla\left(\mathrm{sRic}^\nabla\otimes\omega^{(n-1)}\right) - \mathrm{d}^\nabla R^\nabla\wedge\omega^{(n-2)} - R^\nabla\wedge \mathrm{d}\omega^{(n-2)}\\
&= \nabla\mathrm{sRic}^\nabla\wedge\omega^{(n-1)}\\
&= {*}\nabla\mathrm{sRic}^\nabla ,
\end{aligned}$$

where we have used again Lemma 4.1 and the Bianchi identity $\mathrm{d}^\nabla R^\nabla = 0$. □

Remark 4.6 (i) One can proceed analogously if E is endowed with a Riemannian structure or if E is a complex vector bundle with a Hermitian structure.
(ii) In the considered situation, $\mathrm{sRic}^\nabla = \lambda\,\mathrm{id}_E$ for some $\lambda \in \mathbb{R}$ implies $\mathrm{sRic}^\nabla = 0$. However, if E is a Hermitian vector bundle, it may be interesting to study the analog of the Hermitian Yang–Mills equation.
(iii) Torsion-free symplectic connections with $\nabla\mathrm{sRic}^\nabla = 0$ are investigated in [3,7].

5 First integrals

In the case that M is 4-dimensional, one can ask for self-dual and anti-self-dual solutions of the symplectic Yang–Mills equation

$$\mathrm{d}^\nabla{*}R^\nabla = 0 , \tag{5.1}$$

i.e. for connections $\nabla \in \mathcal{C}(E,\mathsf{b})$ such that

$$*R^\nabla = R^\nabla \tag{5.2}$$

and

$$*R^\nabla = -R^\nabla\ , \tag{5.3}$$

respectively.

Proposition 5.1 *If* $\dim M = 4$*, then*

(i) $*R^\nabla = R^\nabla$ *if and only*

$$R^\nabla = \frac{1}{2}\,\mathrm{sRic}^\nabla \otimes \omega.$$

(ii) $*R^\nabla = -R^\nabla$ *if and only if* $\mathrm{sRic}^\nabla = 0$.

Proof This follows from

$$*R^\nabla = \mathrm{sRic}^\nabla \otimes \omega - R^\nabla.$$

□

The self-duality equation (5.2) as well as the anti-self-duality equation (5.3) possess generalizations to arbitrary dimensions. For the second equation, this is obvious. Namely, by Proposition 5.1, Eq. (5.3) can be generalized by $\mathrm{sRic}^\nabla = 0$. Concerning the self-duality equation, observe that

$$R^\nabla = \frac{1}{n}\,\mathrm{sRic}^\nabla \otimes \omega \tag{5.4}$$

is equivalent to the existence of an endomorphism $L \in \Gamma(\mathrm{End}(E))$ such that $R^\nabla = L \otimes \omega$. Therefore, Eq. (5.4) generalizes Eq. (5.2).

Lemma 5.2 (i) *If* $n = 1$*, then* $R^\nabla = \mathrm{sRic}^\nabla \otimes \omega$ *for any connection* $\nabla \in \mathcal{C}(E)$.

(ii) *If* $n \geq 2$*, then Eq.* (5.4) *is equivalent to*

$$*R^\nabla = \frac{1}{n-1}\,R^\nabla \wedge \omega^{(n-2)}. \tag{5.5}$$

(iii) *It is* $\mathrm{sRic}^\nabla = 0$ *if and only if*

$$*R^\nabla = -R^\nabla \wedge \omega^{(n-2)}. \tag{5.6}$$

Proof The first assertion is trivial. To see the second assertion, suppose that $n \geq 2$ and assume first that Eq. (5.5) holds true. Since

$$*R^\nabla = \mathrm{sRic}^\nabla \otimes \omega^{(n-1)} - R^\nabla \wedge \omega^{(n-2)} \tag{5.7}$$

by Lemma 4.1(iv), this implies

$$\mathrm{sRic}^\nabla \otimes \omega^{(n-1)} = n *R^\nabla\ ,$$

which is equivalent to Eq. (5.4). The converse can be derived straightforwardly. Assertion (iii) is an easy consequence of Eq. (5.7). □

Corollary 5.3 *In case* ω *is a symplectic structure, any connection* $\nabla \in \mathcal{C}(E)$ *such that* $\mathrm{sRic}^\nabla = 0$ *is a solution of Eq.* (5.1)*. If, in addition,* $n \geq 2$*, the same holds true for any connection* $\nabla \in \mathcal{C}(E)$ *that solves Eq.* (5.4).

Proof This follows from the Bianchi identity $\mathrm{d}^\nabla R^\nabla = 0$ and Lemma 5.2. □

Example 5.4 We consider the complex projective space $\mathbb{C}P^n$ endowed with the Fubini–Study symplectic structure

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{D^2} \sum_{k,l=1}^{n} (D\delta_{kl} - \bar{z}_k z_l) \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_l ,$$

where

$$D = 1 + |z_1|^2 + \cdots + |z_n|^2$$

and $z_k = T_k/T_0$ for $k = 1, \ldots, n$ in homogeneous coordinates $[T_0 : \cdots : T_n]$. Let Q be the Hopf bundle, i.e. the U(1)-principal bundle $S^{2n+1} \subset \mathbb{C}^{n+1}$ with U(1)-action

$$((T_0, \ldots, T_n), \lambda) \in S^{2n+1} \times \mathrm{U}(1) \mapsto (T_0\lambda, \ldots, T_n\lambda) \in S^{2n+1}$$

and projection

$$(T_0, \ldots, T_n) \in S^{2n+1} \mapsto [T_0 : \cdots : T_n] \in \mathbb{C}P^n.$$

Then the 1-form

$$Z = \frac{1}{2} \sum_{k=0}^{n} \left(\bar{T}_k \mathrm{d}T_k - T_k \mathrm{d}\bar{T}_k \right)$$

on S^{2n+1} is a connection in Q whose curvature, considered as a 2-form on $\mathbb{C}P^n$ with values in $\mathbb{R}$i, is

$$F^Z = -\mathrm{i}\, \omega_{\mathrm{FS}}. \tag{5.8}$$

Now let $\tau : \mathrm{U}(1) \to \mathrm{Sp}(m, \mathbb{R})$ be any homomorphism into the symplectic group $\mathrm{Sp}(m, \mathbb{R})$ of the vector space $\mathbb{R}^{2m}$ equipped with its standard symplectic structure ω_0. Let E be the real vector bundle $Q \times_\tau \mathbb{R}^{2n}$ associated to Q by means of the representation τ and let b be the almost symplectic structure in E induced by ω_0. Then the connection ∇^Z in E induced by Z lies in the space $\mathcal{C}(E, \mathsf{b})$, and Eq. (5.8) implies that it solves the generalized self-duality equation, i.e.

$$R^{\nabla^Z} = \frac{1}{n} \mathrm{sRic}^{\nabla^Z} \otimes \omega_{\mathrm{FS}}.$$

Examples of torsion-free symplectic connections ∇ that satisfy $\mathrm{sRic}^\nabla = 0$ are constructed in [6].

In order to get a better feeling in handling symplectic connections, in particular symplectic connections with torsion, it is important to examine further classes of examples. There is ample room for future research to build on this study's beginning.

6 Preferred symplectic connections

The aim of this section is to relate the considerations of the previous two sections to the original variational principle suggested by Bourgeois and Cahen. For this, we suppose that ω is a symplectic structure on M. Let $\mathcal{C}_0(M, \omega)$ denote the space of torsion-free symplectic connections. According to Lemma 2.3 and Proposition 2.5, $\mathcal{C}_0(M, \omega)$ is an affine space over the vector space $\mathcal{E}_0^1(M, \omega)$ of all $\theta \in \Omega^1(M, \mathrm{End}(TM))$ such that $\omega(\theta(X)Y, Z)$ is totally symmetric in $X, Y, Z \in \Gamma(TM)$. Let $I_0 : \mathcal{C}_0(M, \omega) \to \mathbb{R}$

be the restriction of the functional I_2 for the case $E = TM$ to the subspace $\mathcal{C}_0(M,\omega)$. The critical points of the functional I_0 are called preferred symplectic connections (cf. [2,7]).

Theorem 6.1 [4] *A connection $\nabla \in \mathcal{C}_0(M,\omega)$ is a critical point of the functional I_0 if and only if*

$$\left(\nabla_X \mathrm{sric}^\nabla\right)(Y,Z) + \left(\nabla_Y \mathrm{sric}^\nabla\right)(Z,X) + \left(\nabla_Z \mathrm{sric}^\nabla\right)(X,Y) = 0$$

for all $X, Y, Z \in \Gamma(TM)$.

For completeness and since, as it seems to us, there is no proof of this result in the literature, we will give a proof of it here.

Let $\sigma : \Omega^1(M,\mathrm{End}(TM)) \to \Omega^1(M,\mathrm{End}(TM))$ be the Bianchi projector. That means that $\sigma(\theta)$ for $\theta \in \Omega^1(M,\mathrm{End}(TM))$ is given by

$$\omega(\sigma(\theta)(X)Y,Z) = \frac{1}{3}\left(\omega(\theta(X)Y,Z) + \omega(\theta(Y)Z,X) + \omega(\theta(Z)X,Y)\right).$$

One easily checks

Lemma 6.2 *For any $\theta_1, \theta_2 \in \Omega^1(M,\mathrm{End}(TM))$,*

$$\omega(\sigma(\theta_1),\theta_2) = \omega(\theta_1,\sigma(\theta_2)).$$

Furthermore, we have

Lemma 6.3 *It is $\mathcal{E}^1_0(M,\omega) = \sigma\left(\mathcal{E}^1(M,\omega)\right)$.*

Proof A direct calculation shows that $\sigma\left(\mathcal{E}^1(M,\omega)\right) \subset \mathcal{E}^1(M,\omega)$. Therefore the assertion is an easy consequence of $\sigma^2 = \sigma$ and

$$\mathcal{E}^1_0(M,\omega) = \left\{\theta \in \mathcal{E}^1(M,\omega) : \sigma(\theta) = \theta\right\}.$$

□

Proof of Theorem 6.1 According to the proof of Theorem 4.5, a connection $\nabla \in \mathcal{C}_0(M,\omega)$ is a critical point of I_0 if and only if

$$\int_M \omega\left(\theta, \nabla \mathrm{sRic}^\nabla\right)\omega^{(n)} = 0$$

for all $\theta \in \mathcal{E}^1_0(M,\omega)$. By Lemma 6.3, the last condition is equivalent to

$$\int_M \omega\left(\sigma(\theta), \nabla \mathrm{sRic}^\nabla\right)\omega^{(n)} = 0$$

for all $\theta \in \mathcal{E}^1(M,\omega)$. By Lemma 6.2, this is the same as

$$\int_M \omega\left(\theta, \sigma\left(\nabla \mathrm{sRic}^\nabla\right)\right)\omega^{(n)} = 0$$

for all $\theta \in \mathcal{E}^1(M,\omega)$. Since $\nabla \mathrm{sRic}^\nabla \in \mathcal{E}^1(M,\omega)$, we obtain that $\nabla \in \mathcal{C}_0(M,\omega)$ is a critical point of I_0 if and only if

$$\sigma\left(\nabla \mathrm{sRic}^\nabla\right) = 0\,,$$

which, because of

$$\omega\left(\left(\nabla_X \mathrm{sRic}^\nabla\right) Y, Z\right) = \left(\nabla_X \mathrm{sric}^\nabla\right)(Y,Z)\,,$$

is the desired relation. □

Remark 6.4 The condition for a connection $\nabla \in \mathcal{C}_0(M,\omega)$ to be preferred can also be expressed as

$$\sigma\left(\delta^\nabla R^\nabla\right) = 0\,,$$

since $\nabla \mathrm{sRic}^\nabla = \delta^\nabla R^\nabla$ as shown in the proof of Theorem 4.5.

We conclude with the following observation.

Proposition 6.5 *Let $n \geq 2$. Then any torsion-free connection $\nabla \in \mathcal{C}(M)$ that solves Eq.* (5.4) *has to be flat.*

Proof Let $\nabla \in \mathcal{C}(M)$ be a torsion-free solution of Eq. (5.4). Then, by the first Bianchi identity, we have

$$\begin{aligned}
0 &= \sum_{i=1}^{n}\left(R^\nabla(\mathsf{e}_i, J^{\mathsf{s}}\mathsf{e}_i)X + R^\nabla(J^{\mathsf{s}}\mathsf{e}_i, X)\mathsf{e}_i + R^\nabla(X,\mathsf{e}_i)J^{\mathsf{s}}\mathsf{e}_i\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\omega(\mathsf{e}_i,J^{\mathsf{s}}\mathsf{e}_i)\mathrm{sRic}^\nabla(X) + \omega(J^{\mathsf{s}}\mathsf{e}_i,X)\mathrm{sRic}^\nabla(\mathsf{e}_i) + \omega(X,\mathsf{e}_i)\mathrm{sRic}^\nabla(J^{\mathsf{s}}\mathsf{e}_i)\right)\\
&= \frac{1}{n}\left(n\,\mathrm{sRic}^\nabla(X) + \mathrm{sRic}^\nabla\left(\sum_{i=1}^{n}(\omega(J^{\mathsf{s}}\mathsf{e}_i,X)\mathsf{e}_i - \omega(\mathsf{e}_i,X)J^{\mathsf{s}}\mathsf{e}_i)\right)\right)\\
&= \frac{n-1}{n}\,\mathrm{sRic}^\nabla(X)\,,
\end{aligned}$$

which yields $\mathrm{sRic}^\nabla = 0$. Thus, by Lemma 5.2, the connection ∇ satisfies Eq. (5.5) as well as Eq. (5.6), and this implies $R^\nabla = 0$. □

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